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An Overview of Self-Consistent Methods for Fiber Reinforced Composites

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Kurt C. Gramoll
Georgia Institute of Technology
Atlanta, Georgia

Alan D. Freed
Lewis Research Center
Cleveland, Ohio

and

Kevin P. Walker
Engineering Science Software, Inc.
Smithfield, Rhode Island

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AN OVERVIEW OF SELF-CONSISTENT METHODS FOR FIBER REINFORCED COMPOSITES

Kurt C. Gramoll*
Georgia Institute of Technology
Atlanta, GA 30332

Alan D. Freed
National Aeronautics and Space Administration
Lewis Research Center
Cleveland, OH 44135

Kevin P. Walker
Engineering Science Software, Inc.
Smithfield, RI 02917

Abstract

The Walker et al. (1989) self-consistent method to predict both the elastic and inelastic effective material properties of composites is examined and compared with the results of other self-consistent and elasticity based solutions. The elastic part of their method is shown to be identical to other self-consistent methods for non-dilute reinforced composite materials; they are the Hill (1965), Budiansky (1965) and Nemat-Nasser et al. (1982) derivations. A simplified form of the non-dilute self-consistent method is also derived.

The predicted, effective, elastic, material properties for a fiber reinforced material using the Walker method was found to deviate from the elasticity solution for the ν_{31} , K_{12} , and μ_{31} material properties (fiber is in the 3 direction) especially at the larger volume fractions. Also, the prediction for the transverse shear modulus, μ_{12} , exceeds one of the accepted Hashin bounds. Only the longitudinal elastic modulus E_{33} agrees with the elasticity solution. The differences between the Walker and elasticity solutions are primarily due to the assumption used in the derivation of the self-consistent method, i.e. the strain fields in the inclusions and the matrix are assumed to remain constant, which is not a correct assumption for a high concentration of inclusions.

*Summer Faculty Fellow at NASA Lewis Research Center.

A summary of some other methods which predict the effective material properties of a composite is also presented. In particular, the two different bounding solutions of Hashin (1965, 1972) for the transverse shear modulus, μ_{12} , are reviewed. Other methods examined include the dilute self-consistent method by Eshelby (1957) and Russel and Acrivos (1972), the Mori-Tanaka method (1973), and the three-phase model of Christensen and Lo (1979b).

Introduction

Over the last 30 years there has been a tremendous effort to predict the effective material properties of a composite material based on the material properties of the individual constituents. The motivation behind this effort is to better design and analyze two part composite materials without actually having to produce and test each and every possible composite material. Recently, Walker et al. (1989) have proposed using a self-consistent method to predict the elastic and inelastic material properties and response. This method was developed for implementation in a finite element code to predict the response of a composite structure. This report will examine in detail the elastic portion of this solution, and will compare it to other self-consistent methods and elasticity solutions to assess its ability to predict the effective material properties of a reinforced composite material.

Some of the original work in developing the self-consistent concept was done by Hershey (1954), Kröner (1958), and Kerner (1956) on single crystal and polycrystalline materials. This work, however, did not examine multiple phase materials such as spherical or cylindrical fiber reinforced composite materials. In 1957, Eshelby introduced his transformation tensor, known as the Eshelby Tensor, which relates the strain of an inclusion constrained inside an infinite elastic matrix to the strain of the same inclusion when placed outside of the matrix without any constraints. This basic tensor has far reaching implications, one of which allows the calculation of the effective material properties of a two phase material. One main assumption in the derivation of the Eshelby Tensor is that multi-inclusions do not effect one another. For low concentrations, this method - known as the self-consistent method for dilute concentration (DSC) - accurately predicts the effective material properties.

Due to the limitation of the DSC method to only low concentrations of inclusions, Hill (1965) and Budiansky (1965) simultaneously developed a self-consistent method that extends the basic Eshelby method to high concentrations - so called the self-consistent method for non-dilute concentration (NSC). This method basically lets the matrix material assume the properties of the desired effective material properties (still unknown), and then places a single inclusion into this matrix. The Eshelby tensor then figures into the description of the strain in both the inclusions and the overall composite. The effect of multiple inclusions is assumed to be modeled by allowing the matrix to have the effective material properties. However, the strain fields in both the inclusion and matrix are still assumed to be constant, as required when using the Eshelby Tensor, which is not valid for densely packed inclusions. The

prediction of elastic properties using NSC method for material representation is better than that of the DSC method. Predictions for the NSC method approach the limiting value of the inclusion properties as the inclusion concentration ratio approaches one; predictions for the DSC method do not. It should also be noted that the solution of the NSC problem is iterative, since the effective material properties are imbedded directly in the formulation.

This paper is mainly concerned with the NSC method, since the solution of Walker et al. (1989) is an extension of this method into the inelastic range. The basic elastic portion of this method is shown to be identical in form to the NSC method. Furthermore, the Nemat-Nasser and Taya (1981) derivation of the self-consistent method is also shown to be the same as the NSC method.

Another method that uses the Eshelby Tensor is the Mori-Tanaka (1973) method. This method accounts for the interaction of the inclusions by defining a second transformation matrix (or strain concentration tensor) between the actual matrix material and the inclusions (recall that the original transformation matrix describes the relation between an inclusion and the effective material properties or homogeneous composite material). By including this additional constraint, the effective material properties will match the limiting values at both low and high concentrations; plus, all five of the elastic constants agree with the elasticity solutions for fibrous composites. However, this method is currently limited to elastic solutions only.

In addition to the self-consistent methods for predicting the effective material properties, there has been a number of elasticity based solutions. Hashin and Shtrikman (1962a, 1962b, 1963) laid much of the basic foundation for employing the elasticity equations using energy methods to develop solutions to the two part spherical and cylindrical inclusion problems. Later, Hashin (1965, 1972, 1979), Hashin and Rosen (1964), and Whitney and Riley (1966) presented exact solutions for most of the material properties. The one exception is the shear modulus of a composite with spherical inclusions, or the transverse shear modulus of a fiber reinforced composite. In these cases, only bounds are obtained. The Hashin bounds are used extensively in the literature to determine what is acceptable, but there is considerable confusion on how to get these bounds and what bounds to use. These bounds are discussed in detail later.

This paper briefly presents each of the basic methods mentioned above so that they can be used to compare results with those of the Walker method. The final section presents Walker's method, and derives many of the basic equations to show its similarity to the other self-consistent methods. A comparison of the various methods is included.

A quick note on the notation. Bold letters represent tensor quantities. The subscripts m and f represent the matrix and fiber (either cylindrical, ellipsoidal, or spherical particulate), respectively. This report assumes the fibers are always oriented in the 3 direction, such that E_{33} is the modulus in the fiber direction. The other two directions, 1 and 2, are perpendicular to the fiber and to each other. Since only

transversely isotropic materials are considered for fiber reinforcing - i.e. the fiber array is hexagonal - the 1 and 2 directions are interchangeable. The five independent material properties used in this report for fiber reinforced composites are: the longitudinal Young modulus, E_{33} , the longitudinal Poisson ratio, ν_{31} , the plain strain transverse bulk modulus, K_{12} , the longitudinal shear modulus, μ_{31} , and the transverse shear modulus, μ_{12} .

Self-Consistent Method for Dilute Concentrations (DSC)

All self-consistent methods rely on two conclusions reached by Eshelby (1957). First, the free strain, ϵ^T , of an inclusion is linearly related by the Eshelby Tensor to the constrained strain, ϵ^c , of the same inclusion imbedded in an elastic body. Second, the strain is uniform in the inclusion. These two conditions allow the effective material properties for dilute concentrations to be determined.

Since the Eshelby tensor is central to the DSC method, it will now be explained using the formulation of Russel (1972, 1973), Christensen (1979a) and Benveniste (1987). Assume that an ellipsoidal shaped body (later to be defined as the inclusion) undergoes a free strain, ϵ^T , without any constraints, i.e. $\sigma_{ij} = 0$. Next, force this inclusion body back to its original ellipsoidal shape by applying tractions at its surface. Now place this ellipsoidal body inside an infinite elastic body that has an elliptical shaped void of the same shape as the original shape of the inclusion. When the inclusion is first placed in the void, just before the tractions are released, the infinite body is strain and stress free. Finally, the tractions on the inclusion are released, causing both the inclusion and body to change shape. At the interface, both the inclusion and the body will have the constraint strain state, ϵ^c . The Eshelby tensor simply relates the free strain, ϵ^T , to this constraint strain, ϵ^c , as

$$\epsilon^c = S\epsilon^T, \quad (1)$$

where S is the Eshelby tensor. The S tensor is a function of the inclusion shape (must be ellipsoidal in shape) and the material properties of the body. Exact solutions for S have been worked out for various inclusion shapes for isotropic and transversely isotropic materials by Eshelby (1957), Mura (1987), Christensen (1979a), Walker (1989), and others.

The Eshelby Tensor can be used in composite analyses where the inclusion has material properties different from those of the constraining body, and where the complete composite undergoes an overall or applied strain, ϵ^o . This is done by modeling the free strain effect as a material property effect. The total strain, ϵ , for an inclusion of the same material as the matrix in which it is embedded equals the overall strain, ϵ^o , plus the constraint strain, ϵ^c , less the original free strain, ϵ^T , giving

$$D_m \epsilon (= \sigma^{-m1}) = D_m (\epsilon^o + \epsilon^c - \epsilon^T), \quad (2)$$

where D_m is the material stiffness matrix for the matrix material. For an inclusion of a different material, say D_f , the strain does not have an original free strain associated with it, since it is modeled as a material property effect. Therefore, the total stress is given by

$$D_f \epsilon (= \sigma^f) = D_f (\epsilon^o + \epsilon^c). \quad (3)$$

Since the stress states for both matrix type inclusion, σ^{m1} , and for the fiber type inclusion, σ^f , must be equal for the same inclusion geometry, Eqs. 2 and 3 can be combined to give

$$D_m (\epsilon^o + \epsilon^c - \epsilon^T) = D_f (\epsilon^o + \epsilon^c), \quad (4)$$

or by using the definition of the inclusion strain, $\epsilon^f = \epsilon^o + \epsilon^c$, Eq. 4 can be rewritten as

$$\epsilon^T = D_m^{-1} (D_m - D_f) \epsilon^f. \quad (5)$$

Substituting Eq. 5 into Eq. 1 gives

$$\epsilon^c = S_m [D_m^{-1} (D_m - D_f)] \epsilon^f, \quad (6)$$

where the components of the Eshelby tensor, S_m , are dependent upon the matrix material's properties. Using the definition of the inclusion strain once again gives

$$\epsilon^f = \epsilon^o + S_m [D_m^{-1} (D_m - D_f)] \epsilon^f. \quad (7)$$

Rearranging Eq. 7 gives

$$\epsilon^f = T \epsilon^o = [I + S_m [D_m^{-1} (D_f - D_m)]]^{-1} \epsilon^o, \quad (8)$$

where T is called the strain concentration tensor, and I is the identity matrix. A second equation is now needed. One that relates the effective material properties, \bar{D} , to the material properties of the matrix and fiber, and to the strain concentration tensor, T .

Following the derivation of Russel and Acrivos (1972), the volume averaged stress in the inclusions and matrix is added together to give the volume averaged stress in an arbitrary subvolume, i.e.

$$\langle \sigma \rangle = \frac{1}{V} \int_{V - \sum_{n=1}^N V_n} \sigma^m dv + \frac{1}{V} \sum_{n=1}^N \int_{V_n} \sigma^f dv, \quad (9)$$

where V is the total volume, V_n is the volume of each inclusion, and $V - \sum V_n$

is the matrix material volume. Also, σ^m and σ^f represent the stress in the matrix and inclusion (or fiber) material, respectively. The first integral is split into two parts, one part representing the total subvolume as a matrix only material, i.e. no inclusions, and a second part representing the inclusions as matrix material. The stresses in each of the three integral terms are then written in terms of strain,

$$\begin{aligned} \langle \sigma \rangle = & \frac{1}{V} \int_V D_m \epsilon^o \, dv - \frac{1}{V} \sum_{n=1}^N \int_{V_n} D_m (\epsilon^o + \epsilon^c) \, dv + \\ & + \frac{1}{V} \sum_{n=1}^N \int_{V_n} D_f (\epsilon^o + \epsilon^c) \, dv. \end{aligned} \quad (10)$$

The strain in the first integral represents the actual strain of the matrix material for the total subvolume as if there are no inclusions. The inclusions are modeled by the second and third integrals. The second integral subtracts out the inclusion volume made of matrix material, D_m , whereas the third integral adds back in the actual inclusion material, D_f . In both cases the strain includes the actual strain, ϵ^o , and the constraint strain, ϵ^c that any inclusion will experience. Equation 10 can be further simplified to

$$\langle \sigma \rangle = D_m \epsilon^o + \frac{1}{V} \sum_{n=1}^N \int_{V_n} (D_f - D_m) \epsilon^f \, dv. \quad (11)$$

The effective material property matrix, \bar{D} , relates the average stress and strain over a representative subvolume, i.e. $\langle \sigma \rangle = \bar{D} \langle \epsilon \rangle$. Assuming that the inclusion strain ϵ^f does not vary over the volume of the inclusion, an assumption that is valid for a single ellipsoidal inclusion, Eshelby (1957), Eq. 11 can be integrated to give

$$\bar{D} \langle \epsilon \rangle = D_m \epsilon^o + V_f (D_f - D_m) \epsilon^f, \quad (12)$$

where $V_f \in [0,1]$ is the volume fraction of the inclusions. Noting that $\langle \epsilon \rangle$ and ϵ^o are the same far from the inclusion, Eq. 12 can be rearranged to give

$$\bar{D} = D_m + V_f (D_f - D_m) T, \quad (13)$$

where $\epsilon^f = T \epsilon^o$.

Combining Eqs. 8 and 13 gives the final form for the effective elastic moduli of the DSC method, i.e.

$$\bar{D} = D_m + V_f (D_f - D_m) [I + S_m [D_m^{-1} (D_f - D_m)]]^{-1}. \quad (14)$$

This equation can be used for both spherical or cylindrical inclusions, the only difference will be in the values for the components of the Eshelby tensor, S_m .

The five effective material properties for a transversely isotropic fiber reinforced composite predicted by Eq. 14 are plotted in Figs. 1-5. In four of the five material properties, the DSC deviates from the elasticity solutions or bounds because of the assumption of no interactions between inclusions. For dilute concentrations, the agreement is reasonable, and this method can be used. The shear modulus and Poisson ratio used for the glass fiber and polyester matrix material properties in the Figs. 1-5 are, $G_m = 0.593$ GPa and $G_f = 29.1$ GPa, and $\nu_m = 0.45$ and $\nu_f = 0.21$ (Richard (1975)).

The Eshelby tensor for a transversely isotropic, continuous, cylindrical fiber reinforced, composite material - such as the one described in the figures - is given by (Walker et al. (1989))

$$S_m = \begin{bmatrix} \frac{5-4\nu_m}{8(1-\nu_m)} & \frac{4\nu_m-1}{8(1-\nu_m)} & \frac{\nu_m}{2(1-\nu_m)} & 0 & 0 & 0 \\ \frac{4\nu_m-1}{8(1-\nu_m)} & \frac{5-4\nu_m}{8(1-\nu_m)} & \frac{\nu_m}{2(1-\nu_m)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3-4\nu_m}{4(1-\nu_m)} \end{bmatrix},$$

where ν_m is the Poisson ratio of the matrix. It should be noted that the three terms in the lower right quadrant are a factor of two larger than the values reported by Eshelby, 1957 and others due to notation. If Voigt notation is used, i.e. S_{ij} , the above values are correct; they are a factor of two larger than those associated with the 4th order tensor notation, i.e. S_{ijkl} . A different Eshelby tensor is required for a spherical, reinforced, composite material.

Mori-Tanaka Method

The Mori-Tanaka method is similar to the DSC method, except it enforces the condition that the effective material properties must match the properties of the inclusion material as the concentration approaches one, i.e. $\bar{D} \rightarrow D_f$ as $V_f \rightarrow 1$. This is accomplished by introducing a new strain concentration tensor, G , which relates the strain in the matrix to the strain in the inclusion, i.e.

$$\epsilon^f = G\epsilon^m, \quad (15)$$

and by rewriting Eq. 13 as

$$\bar{D} = D_m + V_f(D_f - D_m) A, \quad (16)$$

where $\epsilon^f = A\epsilon^o$.

The A matrix is now assumed to include the effects of multiple inclusion interactions, as required for the non-dilute case. Recall, the T matrix only assumed a single inclusion in a matrix with no considerations of multiple inclusion interaction. Combining the definitions of A and G , and eliminating ϵ^f , gives

$$G\epsilon^m = A\epsilon^o. \quad (17)$$

Because the overall strain, ϵ^o , is simply the sum of the relative volume fractions of strains in the matrix and the inclusions, the equation becomes

$$G\epsilon^m = A (V_m\epsilon^m + V_f\epsilon^f), \quad (18)$$

where V_m and V_f are the matrix and fiber volume ratios, respectively, such that $V_m + V_f = 1$. Rearranging Eq. 18, and using Eq. 15 gives

$$A = G [V_m I + V_f G]^{-1}. \quad (19)$$

Equation 19 defines the new strain concentration tensor between the fiber strain ϵ^f and the applied strain ϵ^o . However, G is still not known. Mori and Tanaka (1973) make the assumption that at high concentrations the matrix strain equals the applied strain which lead them to postulate that $G = T$. Making this substitution into Eq. 19, and then substituting back into Eq. 16 gives

$$\bar{D} = D_m + V_f(D_f - D_m) T [V_m I + V_f T]^{-1}, \quad (20)$$

$$\text{where } T = [I + S_m [D_m^{-1} (D_f - D_m)]]^{-1}.$$

This is the final form of the Mori-Tanaka method. It is similar in form to the DSC method, except for the additional term $[V_m I + V_f T]^{-1}$ which acts as a correction factor at non-dilute conditions. At the low and high limits of inclusion concentration, Eq. 20 becomes, respectively,

$$\bar{D} \rightarrow D_m \quad \text{as } V_f \rightarrow 0, \quad \text{and} \quad \bar{D} \rightarrow D_f \quad \text{as } V_f \rightarrow 1.$$

Notice that there is an inconsistency in this derivation; the assumption that $\epsilon^m = \epsilon^o$, implying that $G = T$, when going from Eq. 19 to 20 cannot be

*The theoretical basis for this assumption is discussed in further detail in the "Revised Mori-Tanaka Using the NSC Method " section of this report.

applied earlier in the derivation, say at Eq. 17. Additional comments on the Mori-Tanaka method can also be found in the review papers by Benveniste (1987) and Christensen (1990).

In a more recent paper, Benveniste (1990) has shown that the methods of Willis (1981) and Levin (1976) are equivalent to that of Mori-Tanaka described above. It is worth noting, however, that the P matrix used by Benveniste, defined as $P = S_m D_m^{-1}$, is not symmetric as he states. However, the final effective stiffness matrix \bar{D} is symmetric, as required, even if P is not symmetric.

The five material properties predictions for a transversely isotropic composite using the Mori-Tanaka method, Eq. 20, are plotted in Figs. 1-5 along with other methods that will be discussed in the following sections. The Mori-Tanaka results match the elasticity solutions for 4 of the 5 material properties (Figs 2-5), while the fifth property, the transverse shear modulus, matches the lower Hashin Bound (Fig. 1). The material properties and the Eshelby tensor are the same as those which were used in the DSC predictions. One distinct advantage of this method is that the effective material properties can be determined without an iteration process, which is not true for all methods as will be seen in later sections. Furthermore, the results are in agreement with both the known elasticity solutions and the Hashin bounds, but yet can be written in compact tensor equation form.

Hashin Bounds

One of the first efforts to predict the effective material properties of a two phase composite material, for any concentration, using variational energy principles was done by Hashin (1962), which was later generalized to a multi-phase material by Hashin and Shtrikman (1962a,b and 1963). Note that the Hashin method is not a self-consistent method, but an exact elasticity method. By applying the variational theorems of the theory of elasticity, they determined upper and lower bounds on the effective bulk and shear properties for composites with spherical inclusions. Basically, first they used a general elasticity formulation to obtain the displacement fields of spherical bodies. Then they applied two different boundary conditions; one for strain, and the other for stress. Finally, the theorem of minimum complementary and potential energies were applied to the strain and stress boundary conditions, respectively. Since two different boundary conditions were used, two bounds were determined, an upper bound and a lower bound. For a spherically reinforced composite material, they found the bulk modulus bounds coincided, giving an exact solution, but the bounds for the shear modulus did not coincide.

These same methods were later applied to fiber reinforced, transversely isotropic, composite materials by Hashin (1964). He was able to show that for four of the five properties (i.e. the longitudinal elastic modulus, E_{33} , the longitudinal Poisson ratio, ν_{31} , the plain strain bulk modulus, K_{12} , and the longitudinal shear modulus, μ_{31}), both the upper and lower bounds, due to the minimum potential and complementary energies, coincide giving an exact solution. However just like the sphere model, the

upper and lower bounds for the transverse shear modulus, μ_{12} , do not match. Because there is not an exact solution for μ_{12} , a number of other methods have been developed over the years including the Self-Consistent (Hill, 1965 and Budiansky, 1965), Generalized Self-Consistent (Christensen and Lo, 1979), Mori-Tanaka (Mori and Tanaka, 1973), Modified Mori-Tanaka (Luo and Weng, 1987), and Differential methods (Roscoe 1952). Although these other methods can also predict all material properties, this report concentrates on the ambiguous μ_{12} term.

Because of the strong theoretical basis of the Hashin bounds, all other methods are generally compared with it as a check for correctness. If a particular method does not match the Hashin bounds for the four known properties, or falls outside of the upper or lower bounds for the fifth property, μ_{12} , then the method is considered to be in error. However upon reviewing the literature, one can very quickly become confused with the many different bounds that are referred to as the 'Hashin Bounds'. The main cause for this confusion is that Hashin actually derived two sets of bounds for a fiber reinforced composite with randomly placed parallel fibers of different sizes. Hashin (1979) also developed a third set of bounds for a composite constructed with identical fibers arranged in a hexagonal pattern which is not discussed in this paper.

The original Hashin bounds for μ_{12} that were set forth in his 1965 paper required the solution of 8 linear equations. However, if the fibers are assumed to be solid (not hollow) the number of equations reduce to 6, as was presented by Hashin (1972) in a later paper. The upper (+) and lower (-) bounds, based on the potential and complementary energies, respectively, are

$$\mu_{12}^+ = \mu_m \left[1 - \frac{2(1-\nu_m)}{1-2\nu_m} V_f A_4^c \right], \quad (21)$$

$$\mu_{12}^- = \frac{\mu_m}{\left[1 - \frac{2(1-\nu_m)}{1-2\nu_m} V_f A_4^\sigma \right]}, \quad (22)$$

where ν_m = Poisson's ratio of matrix,

μ_m = shear modulus of matrix,

V_f = fiber volume fraction, and

A_4 = constant from the solution of the 6 linear equations.

The 6 linear equations, in matrix form, that must be solved to determine the constants A_4^c are

$$\begin{bmatrix}
1 & \frac{1}{V_f} & V_f^2 & V_f & 0 & 0 \\
0 & -\frac{3-4\nu_m}{3-2\nu_m} \cdot \frac{1}{V_f} & -2V_f^2 & \frac{V_f}{1-2\nu_m} & 0 & 0 \\
1 & 1 & 1 & 1 & -1 & -1 \\
0 & -\frac{3-4\nu_m}{3-2\nu_m} & -2 & \frac{1}{1-2\nu_m} & 0 & \frac{3-4\nu_f}{3-2\nu_f} \\
1 & \frac{3}{3-2\nu_m} & -3 & \frac{1}{1-2\nu_m} & -\frac{\mu_f}{\mu_m} & \frac{-3}{3-2\nu_f} \cdot \frac{\mu_f}{\mu_m} \\
0 & \frac{-1}{3-2\nu_m} & 2 & \frac{-1}{1-2\nu_m} & 0 & \frac{1}{3-2\nu_f} \cdot \frac{\mu_f}{\mu_m}
\end{bmatrix}
\begin{Bmatrix}
A_1^e \\
A_2^e \\
A_3^e \\
A_4^e \\
A_5^e \\
A_6^e
\end{Bmatrix}
=
\begin{Bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{Bmatrix} \quad (23)$$

and for A_4^σ are

$$\begin{bmatrix}
1 & \frac{3}{3-2\nu_m} \cdot \frac{1}{V_f} & -3V_f^2 & \frac{V_f}{1-2\nu_m} & 0 & 0 \\
0 & \frac{-1}{3-2\nu_m} \cdot \frac{1}{V_f} & 2V_f^2 & \frac{-V_f}{1-2\nu_m} & 0 & 0 \\
1 & 1 & 1 & 1 & -1 & -1 \\
0 & -\frac{3-4\nu_m}{3-2\nu_m} & -2 & \frac{1}{1-2\nu_m} & 0 & \frac{3-4\nu_f}{3-2\nu_f} \\
1 & \frac{3}{3-2\nu_m} & -3 & \frac{1}{1-2\nu_m} & -\frac{\mu_f}{\mu_m} & \frac{-3}{3-2\nu_f} \cdot \frac{\mu_f}{\mu_m} \\
0 & \frac{-1}{3-2\nu_m} & 2 & \frac{-1}{1-2\nu_m} & 0 & \frac{1}{3-2\nu_f} \cdot \frac{\mu_f}{\mu_m}
\end{bmatrix}
\begin{Bmatrix}
A_1^\sigma \\
A_2^\sigma \\
A_3^\sigma \\
A_4^\sigma \\
A_5^\sigma \\
A_6^\sigma
\end{Bmatrix}
=
\begin{Bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{Bmatrix} \quad (24)$$

These two bounds, referred to as the composite cylinder assemblage (CCA) bounds by Hashin, are plotted in Fig. 6 as a function of the fiber volume fraction. In later papers, Hashin (1972, 1979) presented the following closed form solutions to Eqs. 21 and 22 for the 6 linear equations; they are

$$\mu_{12}^+ = \mu_m \left[1 + \frac{(1+\beta_m)V_f}{\rho - V_f \left[1 + \frac{3\beta_m^2(1-V_f)^2}{\alpha V_f^3 + 1} \right]} \right], \quad (25)$$

$$\mu_{12}^- = \mu_m \left[1 + \frac{(1+\beta_m)V_f}{\rho - V_f \left[1 + \frac{3\beta_m^2(1-V_f)^2}{\alpha V_f^3 - \beta_m} \right]} \right] \quad (26)$$

$$\text{where } \alpha = \frac{\beta_m - \gamma\beta_f}{1 + \gamma\beta_f}, \quad \rho = \frac{\gamma + \beta_m}{\gamma - 1},$$

$$\beta_m = \frac{K_m^*}{K_m^* + 2\mu_m}, \quad \beta_f = \frac{K_f^*}{K_f^* + 2\mu_f},$$

$$\gamma = \frac{\mu_f}{\mu_m}, \text{ and}$$

K_m^* and K_f^* = transverse bulk moduli for the matrix and fiber.

There has been considerable confusion in the literature regarding the CCA bounds, because of the two different forms of these equations. Unless one reads Hashin's papers carefully, it is easy to assume that Eqs. 21 and 22, and Eqs. 25 and 26 are two different sets of bounds; they are not. Furthermore, if one uses the standard definition for the bulk modulus,

$$K = \frac{2\mu(1+\nu)}{3(1-2\nu)} \text{ or } K = \lambda + 2\mu/3, \quad (27)$$

for either the matrix or fiber material, Eqs. 21 and 22 will give different results from those of Eqs. 25 and 26. Hashin, as well as others, e.g. Christensen (1979a), define K^* as the plain strain or transverse bulk modulus, which is different than the standard isotropic bulk modulus, K . The K^* term originated from the transversely isotropic, engineering, material property K_{12} , which is defined as $K_{12} = \frac{1}{2}(C_{11} + C_{12})$ where C_{ij} are the stiffness moduli. This transversely isotropic definition is then carried over to describe the bulk modulus of both the isotropic fiber and matrix materials. Therefore, the correct definition to use with Eqs. 25 and 26 is

$$K^* = \lambda + \mu = \frac{\mu}{1-2\nu} = \frac{\mu^2}{3\mu-E}. \quad (28)$$

To make matters more confusing, Hashin (1965) developed a second set of upper and lower bounds for μ_{12} using the Polarization Extremum Principle that was developed by Hashin and Shtrikman (1962a,b) and Hill (1965) for spherical inclusions. A detailed derivation can also be found in Hashin (1972). The final upper and lower bounds for the transverse shear modulus of this method are

$$\mu_{12}^+ = \mu_m + \frac{1 - V_f}{\frac{1}{\mu_f - \mu_m} + \frac{(K_f^* + 2\mu_f)V_f}{2\mu_f(K_f^* + \mu_f)}}, \quad (29)$$

$$\mu_{12}^- = \mu_f + \frac{V_f}{\frac{1}{\mu_f - \mu_m} + \frac{(K_m^* + 2\mu_m)(1 - V_f)}{2\mu_m(K_m^* + \mu_m)}}, \quad (30)$$

where: μ_m, μ_f = shear moduli of the matrix and fiber,
 K_m^*, K_f^* = transverse bulk moduli of the matrix and fiber, and
 V_f = fiber volume fraction.

These equations, referred to as the Arbitrary Cylindrical Phase (ACP) model, are plotted in Fig. 6 along with the non-dilute self-consistent method (NSC). Note that they do not coincide with the previous bounds, Eqs. 21 and 22, or Eqs. 25 and 26. Because the two sets of bounds - the CCA and ACP - do not coincide, they tend to be confused with one another. In fact, both bounds are still being used and both are labeled as the 'Hashin Bounds'. Hashin (1979) summarized both sets of bounds briefly and stated that the upper bound of the ACP method along with the lower bound of the CCA method should be used. This gives the tightest bounds for the unknown transverse shear modulus μ_{12} , and are the Hashin bounds plotted in Fig. 1.

Three-Phase Model

The Hashin model discussed above gave the exact solution for 4 of the 5 effective material properties, but only the bounds for the remaining property, the transverse shear modulus. Christensen and Lo (1979) later developed an exact solution for the transverse shear modulus by using a three phase geometry. Like the Hashin solution, they use the basic elasticity solution for cylindrical geometries, and thus eliminate the need to use the Eshelby solution for inclusions. Although the three-phase model is not a self-consistent model, it is often referred to as the generalized self-consistent model because it uses three phases, including an outer phase that is modeled as an equivalent, homogeneous or smeared, composite material similar to the other self-consistent methods.

The geometry used in this model consists of a cylindrical fiber embedded in a larger cylindrical matrix, which is placed in an infinite, homogeneous, composite material that represents the effective composite material. The method starts by assuming the solution for the displacements in all three phases, and then the eight boundary conditions are applied. The displacements, u_r and u_θ , in the outer phase are

$$u_{re} = \frac{b}{4\mu} \left[\frac{2r}{b} + a_3(\eta+1)\frac{b}{r} + c_3\left(\frac{b}{r}\right)^3 \right] \cos(2\theta), \quad (31)$$

$$u_{\theta e} = \frac{b}{4\mu} \left[\frac{-2r}{b} - a_3(\eta-1)\frac{b}{r} + c_3\left(\frac{b}{r}\right)^3 \right] \sin(2\theta), \quad (32)$$

where b is the radius to the outer edge of the matrix material, and $r \geq b$. In the matrix phase, the displacements are

$$u_{rm} = \frac{b}{4\mu_m} \left[a_2(\eta_m-3)\left(\frac{r}{b}\right)^3 + d_2\frac{r}{b} + c_2(\eta_m+1)\frac{b}{r} + b_2\left(\frac{b}{r}\right)^3 \right] \cos(2\theta), \quad (33)$$

$$u_{\theta m} = \frac{b}{4\mu_m} \left[a_2(\eta_m-3)\left(\frac{r}{b}\right)^3 - d_2\frac{r}{b} - c_2(\eta_m-1)\frac{b}{r} + b_2\left(\frac{b}{r}\right)^3 \right] \sin(2\theta), \quad (34)$$

where $a \leq r \leq b$. And in the fiber phase, they are given by

$$u_{rf} = \frac{b}{4\mu_f} \left[a_1(\eta_f-3)\left(\frac{r}{b}\right)^3 + d_1\frac{r}{b} \right] \cos(2\theta), \quad (35)$$

$$u_{\theta f} = \frac{b}{4\mu_f} \left[a_1(\eta_f+3)\left(\frac{r}{b}\right)^3 - d_1\frac{r}{b} \right] \sin(2\theta), \quad (36)$$

where $r \leq a$. The constants a_3 , c_3 , a_2 , d_2 , c_2 , b_2 , a_1 , and d_1 are to be determined, and $\eta = 3-4\nu$, $\eta_m = 3-4\nu_m$, and $\eta_f = 3-4\nu_f$.

In addition to the eight boundary conditions, the energy requirement (Eshelby, 1956) that

$$\int_0^{2\pi} \left[\sigma_r^o u_{re} + \tau_{r\theta}^o u_{\theta e} - \sigma_{re}^o u_r^o - \tau_{r\theta e}^o u_\tau^o \right]_{r=b} b d\theta = 0 \quad (37)$$

must also be satisfied. The final result for the effective transverse shear modulus is given by the quadratic equation

$$(\mu_{12}/\mu_m)^2 A + (\mu_{12}/\mu_m) B + D = 0 \quad (38)$$

where

$$A = 3V_f(1-V_f)^2[\mu_f/\mu_m-1][\mu_f/\mu_m+\eta_f] + \\ [(\mu_f/\mu_m)\eta_m + \eta_f\eta_m - ((\mu_f/\mu_m)\eta_m - \eta_f)V_f^3] \times \\ [\eta_m V_f(\mu_f/\mu_m-1) - ((\mu_f/\mu_m)\eta_m + 1)], \quad (39)$$

$$B = -6V_f(1-V_f) [\mu_f/\mu_m-1][\mu_f/\mu_m+\eta_f] + [(\mu_f/\mu_m)\eta_m + (\mu_f/\mu_m-1)V_f + 1] \times \\ [(\eta_m-1)(\mu_f/\mu_m+\eta_f) - 2V_f^3((\mu_f/\mu_m)\eta_m - \eta_f)] + \\ V_f(\eta_m+1)[\mu_f/\mu_m-1][\mu_f/\mu_m + \eta_f + ((\mu_f/\mu_m)\eta_m - \eta_f)V_f^3], \quad (40)$$

$$D = 3V_f(1-V_f)^2[\mu_f/\mu_m-1][\mu_f/\mu_m+\eta_f] + [(\mu_f/\mu_m)\eta_m + (\mu_f/\mu_m-1)V_f + 1] \times \\ [\mu_f/\mu_m + \eta_f + ((\mu_f/\mu_m)\eta_m - \eta_f)V_f^3]. \quad (41)$$

The quadratic equation, Eq. 38, will have one negative root and one positive root, which is the effective transverse shear modulus, μ_{12} .

The positive root of Eq. 38, plotted in Fig. 1, lies between both sets of Hashin bounds which demonstrates the validity of the solution. The one assumption made in the derivation of the Christensen-Lo model is that all fibers interact similarly with one another, and thus the outer phase can be modeled as an equivalent composite material. Even though this is a sound physical model, it is still an assumption that is required to solve the problem, and should be stated. Nevertheless, we shall refer to their result as the exact elasticity solution. The difficulty in using the three-phase model is that, like the Hashin model, it requires solving an elasticity problem. This makes it very difficult, if not impossible, to extend the method into the inelastic domain. In the self-consistent methods, the problem is formulated using the strains in tensor notation, which lends itself nicely to including inelastic strains.

Self-Consistent Method for Non-Dilute Concentrations (NSC)

In order to extend the basic self-consistent method to non-dilute concentrations of inclusions, Hill (1965) and Budiansky (1965) used the idea of an equivalent matrix surrounding the inclusion. This equivalent or homogeneous matrix was assumed to have the material properties of the overall composite, and thus accounts for the effect of the inclusions interacting with one another.

The final form of Hill and Budiansky for predicting the effective material properties can be easily derived by starting with the results obtained in the DSC method. First, recall Eq. 4, and modify the D_m tensor to be the \bar{D} tensor which represents the effective material properties (remember, the fiber is now assumed to be embedded in a homogeneous composite material \bar{D} , and not into the matrix D_m).

$$\bar{D} (\epsilon^o + \epsilon^c - \epsilon^T) = D_f (\epsilon^o + \epsilon^c) \quad (42)$$

Preceding in the same manner as with the DSC method, as outlined in Eqs. 5 through 8, the final strain concentration tensor will be

$$\bar{T} = [I + \bar{S} [\bar{D}^{-1} (D_f - \bar{D})]]^{-1}. \quad (43)$$

This can now be used in Eq. 13 to give

$$\bar{D} = D_m + V_f(D_f - D_m) [I + \bar{S} [\bar{D}^{-1} (D_f - \bar{D})]]^{-1}, \quad (44)$$

which is the standard form of the NSC method. It should be noted also that the Eshelby tensor, \bar{S} , is now a function of the homogenized composite phase. If the fibers are arranged in a hexagonal array, the composite will be transversely isotropic and the \bar{S} tensor will have the following components (Walker et al. 1989)

$$\bar{S} = \begin{bmatrix} \frac{5\bar{D}_{11}+\bar{D}_{12}}{8\bar{D}_{11}} & \frac{3\bar{D}_{12}-\bar{D}_{11}}{8\bar{D}_{11}} & \frac{\bar{D}_{13}}{2\bar{D}_{11}} & 0 & 0 & 0 \\ \frac{3\bar{D}_{12}-\bar{D}_{11}}{8\bar{D}_{11}} & \frac{5\bar{D}_{11}+\bar{D}_{12}}{8\bar{D}_{11}} & \frac{\bar{D}_{13}}{2\bar{D}_{11}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3\bar{D}_{11}-\bar{D}_{12}}{4\bar{D}_{11}} \end{bmatrix}. \quad (45)$$

While Eq. 44 is not in the same form as the equations given by Hill (1965) and Budiansky (1965), it can be rearranged and manipulated to match their solutions. Nemat-Nasser (1981) has also derived an alternate form of Eq. 44, i.e.

$$\bar{D} = D_m + V_f(D_f - D_m) (\bar{D} - D_f)^{-1} \bar{D} [(\bar{D} - D_f)^{-1} \bar{D} - \bar{S}]^{-1}, \quad (46)$$

where his notation has been changed to match the notation in this paper. Equation 46 can be manipulated as follows to produce Eq. 44.

$$\bar{D} = D_m + V_f(D_f - D_m) (\bar{D} - D_f)^{-1} [(\bar{D} - D_f)^{-1} - \bar{S} \bar{D}^{-1}]^{-1},$$

$$\bar{D} = D_m + V_f(D_f - D_m) [I - \bar{S} \bar{D}^{-1} (\bar{D} - D_f)]^{-1},$$

$$\bar{D} = D_m + V_f(D_f - D_m) [I + \bar{S} [\bar{D}^{-1} (D_f - \bar{D})]]^{-1}. \quad (\text{Same as 44})$$

Walker et al. (1989) have also developed an expression for the elastic effective material properties independent of the above methods. But as will be seen, their method is also another form of the basic NSC method. Their equation for the elastic effective material properties of a composite is given by Eq. 165 in Walker et al. (1989); it is

$$\begin{aligned} \bar{D} = & V_f D_f [I + \bar{S} [\bar{D}^{-1} (D_f - \bar{D})]]^{-1} + \\ & + (1-V_f) D_m [I + \bar{S} [\bar{D}^{-1} (D_m - \bar{D})]]^{-1}. \end{aligned} \quad (47)$$

This form is difficult to manipulate to give Eq. 44. But if one returns to Eq. 114 in their report from which Eq. 47 was derived, an alternate derivation will give Eq. 44. Equation 114 of Walker - without the inelastic strain (eigenstrain) terms - is

$$\Delta\sigma^o = V_f D_f \Delta\epsilon^T(f) + V_m D_m \Delta\epsilon^T(m), \quad (48)$$

which is the rule of mixtures for the stresses in an elastic composite. Furthermore, one observes that Eq. 48 is the incremental form of Eq. 9. Therefore, following the same method of derivation, Eq. 48 becomes Eq. 13,

$$\bar{D} = D_m + V_f(D_f - D_m) \bar{T}, \quad (13)$$

where T is replaced by \bar{T} . It is interesting to note that Eq. 13 is also the same as Eq. 87 in Walker et al. (1989). The elastic portion of that equation is given by

$$\Delta\sigma^o = D_m \Delta\epsilon^o - \frac{1}{V} \int_V \delta D(r) \Delta\epsilon^T(r) dv, \quad (49)$$

where V is the total volume of the composite subvolume, and δD is equal to 0 in the matrix region and equal to $D_f - D_m$ in the fiber region. Rewriting Eq. 49 in the standard notation used in this paper, and evaluating the integral gives

$$\Delta\sigma^o = D_m \Delta\epsilon^o - 0 - V_f [D_m \Delta\epsilon^T(f) - D_f \Delta\epsilon^T(f)]. \quad (50)$$

This can be rearranged to give

$$\bar{D} \Delta\epsilon^o = D_m \Delta\epsilon^o + V_f(D_f - D_m) \Delta\epsilon^T(f). \quad (51)$$

This too is the same as Eq. 13, except that it is in an incremental form. Thus both Eqs. 87 and 114 of Walker's are the same, and are also the same as Eq. 13 in this report. The only other term that is needed is the strain concentration tensor \bar{T} . Again turning to the work of Walker et al. (1989), they derive this tensor for the inelastic case though extensive equation manipulation using a Green's function formulation. The end result, given as Eq. 125 of their report, produces a \bar{T} tensor identical to that in Eq. 43 of this report. In summary, the elastic portion of their self-consistent method is the same as the standard NSC method.

As expected, both Eq. 44 (the basic NSC equation) and Eq. 47 (the Walker solution) give the same results, but Eq. 44 converges much more rapidly. The results of the NSC method are shown in Figs. 1-6. In all cases, except for E_{33} , the NSC results deviated from the elasticity solutions. This can be partially attributed to the assumption of constant strain in the inclusions, which is not true at high concentrations of inclusions.

Useful insight can be acquired by comparing Budiansky's (1965) equations with Eq. 44 for the bulk and shear moduli of a composite where the inclusions are spherical voids. Budiansky derived elastic solutions which show that if the bulk and shear moduli of the inclusions are zero valued (voids), and the matrix bulk modulus is infinite (incompressible), then the shear and bulk moduli of the composite will be given by

$$\bar{\mu} = \left[\frac{3(1-2V_f)}{3-V_f} \right] \mu_m, \quad (52)$$

$$\bar{K} = \left[\frac{4(1-2V_f)(1-V_f)}{V_f(3-V_f)} \right] K_m. \quad (53)$$

Equation 44, which is in tensor form, can be rearranged to give Eq. 52, since there are no off-diagonal shear terms in the D matrix. However, Eq. 53 cannot be obtained because the bulk modulus K is a combination of two or more D components, and can therefore only be solved numerically. But when Eq. 44 is solved by successive substitution with the conditions $\mu_f = K_f = 0$ and $K_m = \infty$, the solution diverges from the correct answer as given in Eqs. 52 and 53. One reason for this can be seen if $D_f = 0$ is substituted into Eq. 44 to give

$$\bar{D} = D_m - V_f D_m [I + S_m [\bar{D}^{-1} (-\bar{D})]]^{-1}, \quad (54)$$

or simplifying,

$$\bar{D} = D_m [I - V_f [I - S_m]^{-1}]. \quad (55)$$

The $[I-S]$ term is singular whenever $\nu = 0.5$ is used to calculate the Eshelby tensor, S_m . Also, D_m is very large since $K_m = \infty$. These two conditions make it extremely hard to converge to the correct answer

numerically. Another numerical difficulty that arises in solving Eq. 54 is when the Poisson ratio, ν , is less than 0.2, which can easily happen during the convergence process. If $\nu < 0.2$ or $\nu > 1.0$, then the Eshelby tensor will have negative terms in the main diagonal of the matrix. This makes all direct substitution solution techniques diverge.

The above two difficulties are not directly apparent for the case of fiber inclusions because of the complexity of the equations, but they may be one cause for the slow convergence. It should be noted that the \bar{D} matrix does become symmetrical after the iteration process has fully converged, thereby satisfying the requirement of a symmetric stiffness matrix.

Revised Mori-Tanaka using the NSC Method

Recall, in the derivation of the Mori and Tanaka (1973) method a new strain concentration matrix, G , was introduced which relates the strain in the inclusion to the strain in the matrix ($\epsilon^f = G \epsilon^m$). This was then used to derive the following relation

$$\bar{D} = D_m + V_f(D_f - D_m) A, \quad (56)$$

$$\text{where } A = G [V_m I + V_f G]^{-1}.$$

At this point, Mori-Tanaka method sets $G = T$ using ad hoc arguments that they advocate are a consequence of considering the non-dilute case. However, the actual form of T is still derived from the dilute case. This can be shown by deriving the strain concentration matrix, G , in terms of the non-dilute condition. From Eq. 47 (or Eqs. 125 and 142 of Walker et al., 1989) the strain in the fiber and matrix for the non-dilute case can be written as

$$\epsilon^f = [I + \bar{S} [\bar{D}^{-1} (D_f - \bar{D})]]^{-1} \epsilon^o = T_f \epsilon^o \quad (57)$$

$$\epsilon^m = [I + \bar{S} [\bar{D}^{-1} (D_m - \bar{D})]]^{-1} \epsilon^o = T_m \epsilon^o \quad (58)$$

Substituting Eqs. 57 and 58 into $\epsilon^f = G \epsilon^m$ gives

$$T_f \epsilon^o = G T_m \epsilon^o, \quad (59)$$

and rearranging gives

$$G = T_f T_m^{-1}. \quad (60)$$

Looking at the limiting case as $V_f \rightarrow 0$, then $\bar{D} \rightarrow D_m$ and $\bar{S} \rightarrow S_m$, and G becomes

$$G \rightarrow G_m = [I + S_m [D_m^{-1} (D_f - D_m)]]^{-1} \equiv T. \quad (61)$$

However, for the other limiting case as $V_f \rightarrow 1$, then $\bar{D} \rightarrow D_f$ and $\bar{S} \rightarrow S_f$, and G becomes

$$G \rightarrow G_f = I + S_f [D_f^{-1} (D_m - D_f)] \equiv T, \quad (62)$$

which is not the same T as implied in the original Mori-Tanaka method. Therefore, the Mori-Tanaka method cannot assume $G = T$ for all cases of V_f .

It is interesting to note that when G , as derived in Eq. 60, is used in Eq. 56, the results for the effective material properties, \bar{D} , are identical to the results from the NSC method. This should not be too surprising however, since the new G was derived using the same basic equations (Eqs. 47) for the matrix and fiber strain. The reduction of Eq. 56 to the NSC equation is accomplished by substituting G back into Eq. 56 giving

$$A = T_f T_m^{-1} [V_m I + V_f T_f T_m^{-1}]^{-1}, \quad (63)$$

which can be simplified to

$$A = T_f [V_m T_m + V_f T_f]^{-1}. \quad (64)$$

The T terms in the brackets can be replaced by Eqs. 57 and 58, giving

$$A = T_f \epsilon^0 [V_m \epsilon^m + V_f \epsilon^f]^{-1}. \quad (65)$$

This reduces to

$$A = T_f = [I + \bar{S} [\bar{D}^{-1} (D_f - \bar{D})]]^{-1}, \quad (66)$$

because $\epsilon^0 = V_m \epsilon^m + V_f \epsilon^f$, which is the same as the NSC method. Therefore, this modified Mori-Tanaka method is just the NSC method.

Summary and Conclusion

Various self-consistent and elasticity solutions have been presented and compared with the method developed by Walker et al. (1989) for predicting the effective material properties. It was found that the Walker et al. method for the elastic material properties is identical to the self-consistent method for non-dilute concentrations (NSC) presented by Hill (1965) and Budiansky (1965), and later by Nemat-Nasser (1981). Also, the solutions for the NSC method are shown to deviate from the elastic solutions of Hashin (1965, 1972) and Christensen and Lo (1979), and the Mori-Tanaka (1973) self-consistent method, when the fiber volume ratio V_f

exceeded about one-half. However, for practical metal matrix composites where the fiber volume ratio is generally less than one-half, all methods give adequate results.

Because the NSC method does not stay within the accepted bounds for the effective elastic material properties, it should only be used with great care and caution. A second detriment to using the NSC method is the iterative nature of the solution. The convergence is slow using successive substitution, and would be difficult to implement into a finite element program where this method would be used at every element node point. It might be possible to use a higher order solution technique to improve on the convergence, but that was not investigated in this report.

In order to stay within the elastic solution bounds, and to minimize the solution time, it is recommended that a non-iterative approach such as the Mori-Tanaka or Christensen-Lo methods be examined for extension to include the inelastic material properties. The Christensen-Lo method would be the first choice because it is an exact elasticity solution, but it will be the more difficult of the two to extend into the inelastic region. The other choice, the Mori-Tanaka method, should be easier to extend because it is already in tensor form, but there are still unanswered questions about its theoretical validity.

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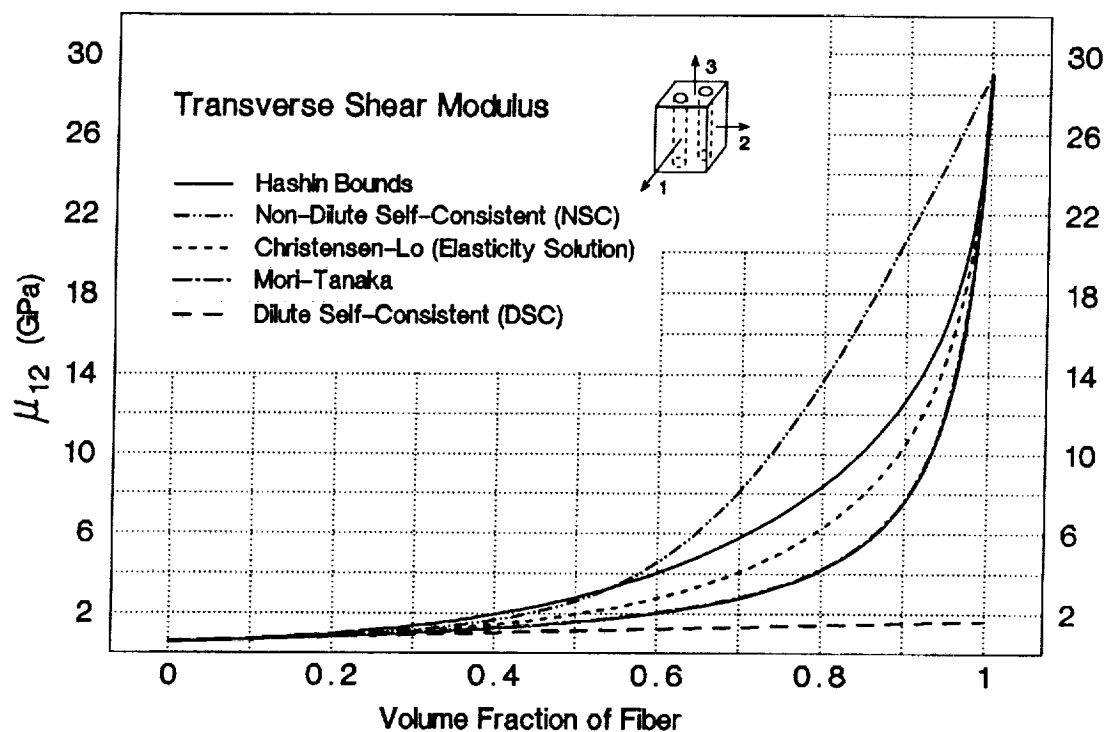


Fig. 1. Transverse Shear Modulus for a Glass/Polyester Fiber Composite.

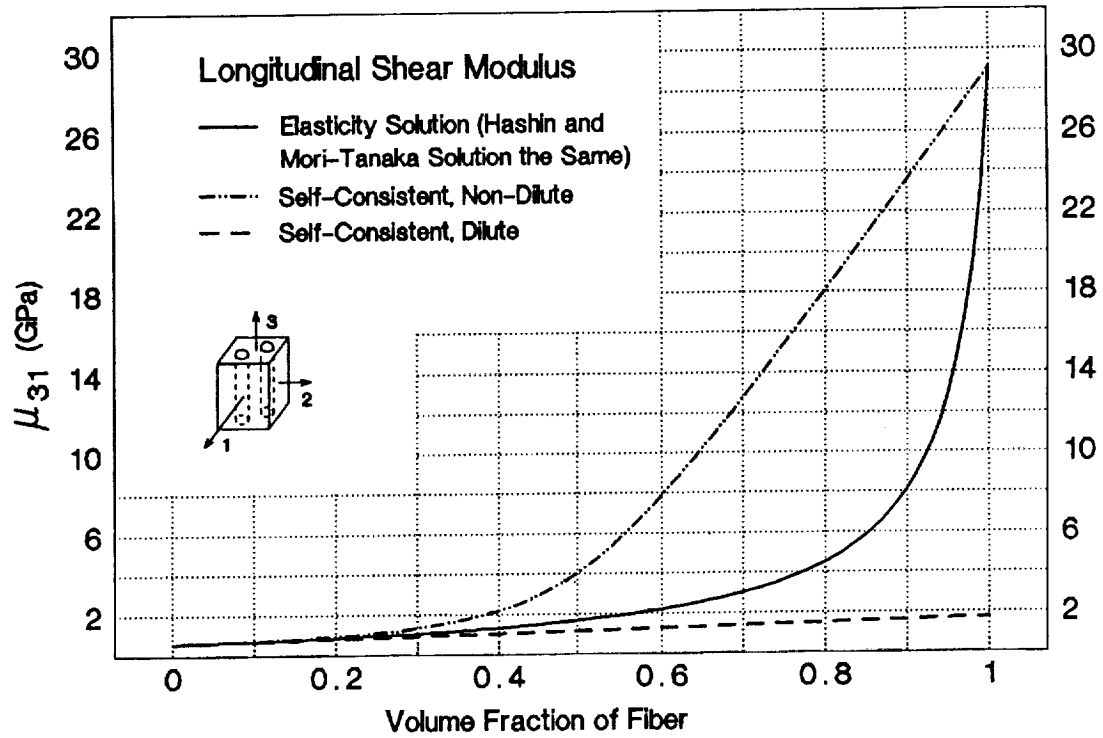


Fig. 2. Longitudinal Shear Modulus for a Glass/Polyester Fiber Composite.

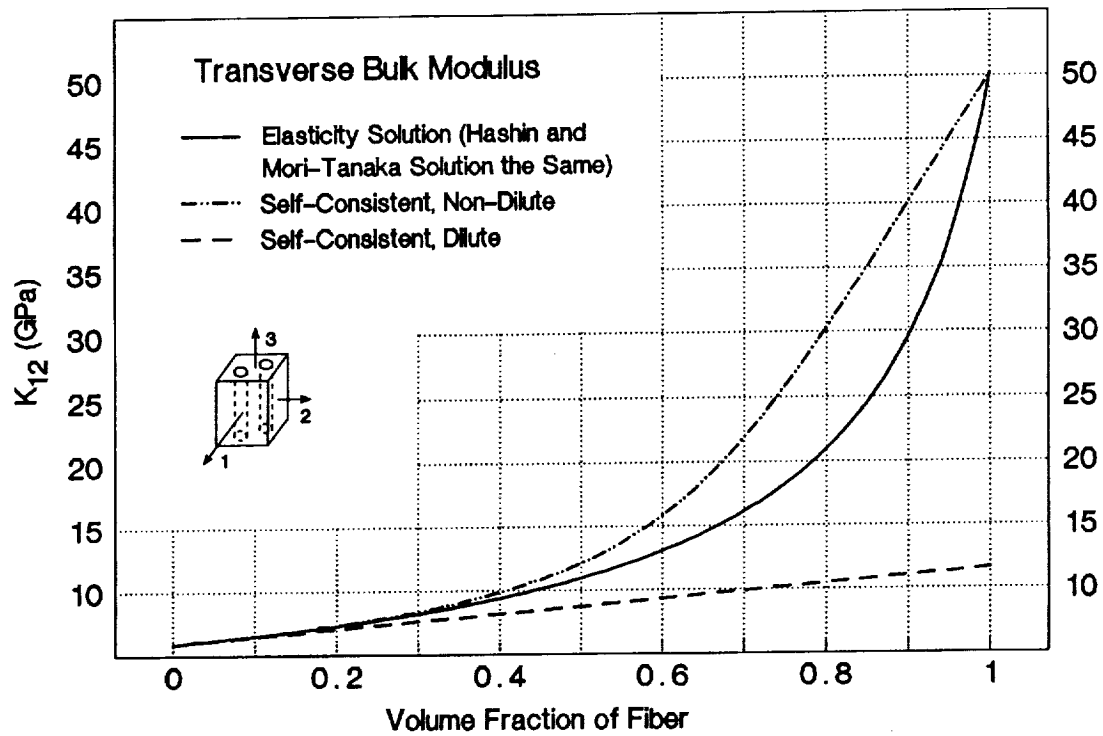


Fig. 3. Transverse Bulk Modulus for a Glass/Polyester Fiber Composite.

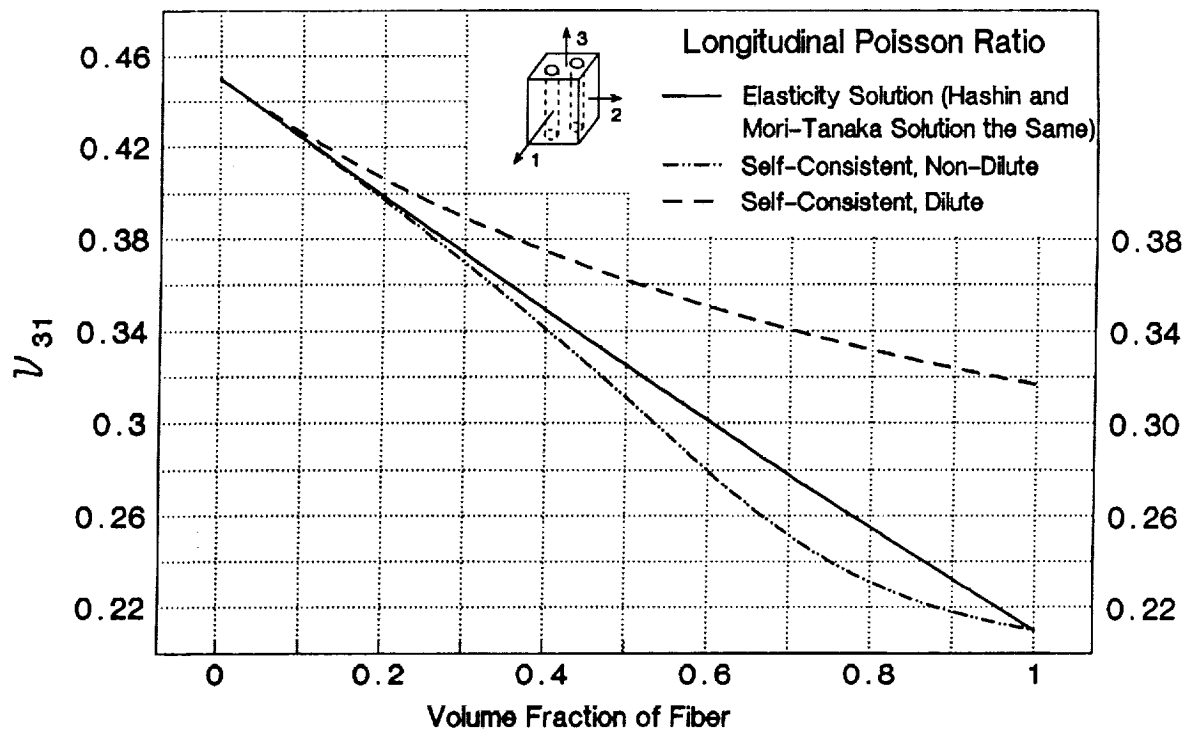


Fig. 4. Longitudinal Poisson Ratio for a Glass/Polyester Fiber Composite.

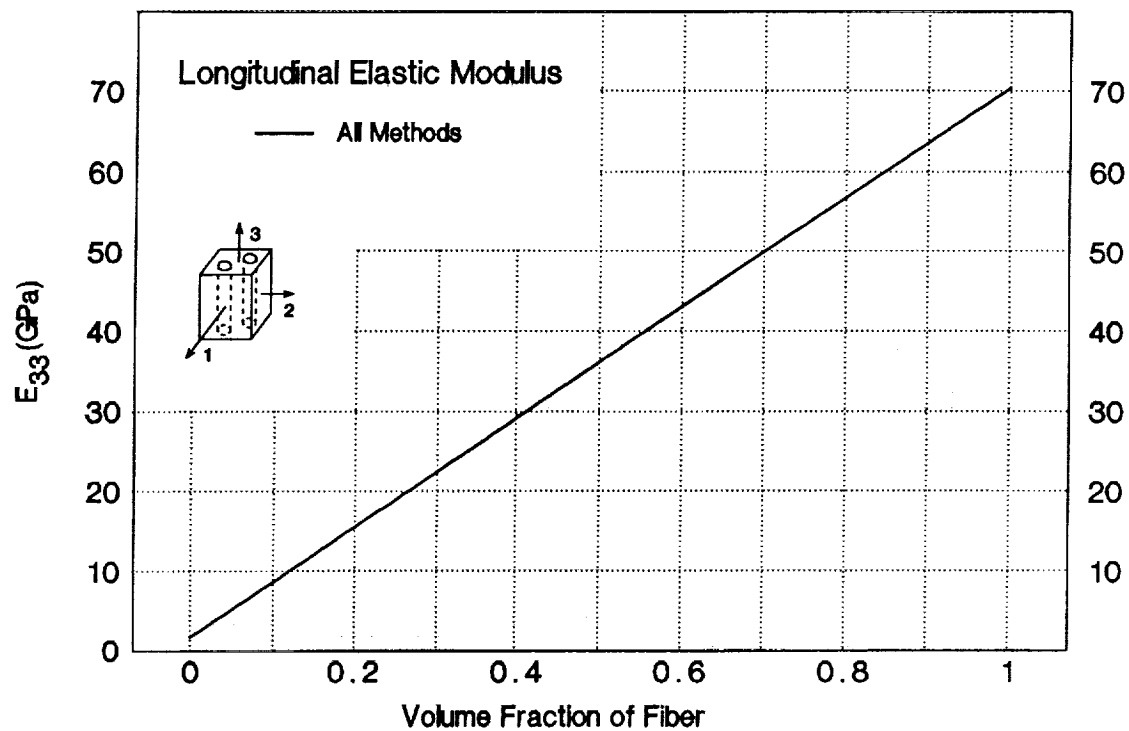


Fig. 5. Longitudinal Elastic Modulus for a Glass/Polyester Fiber Composite.

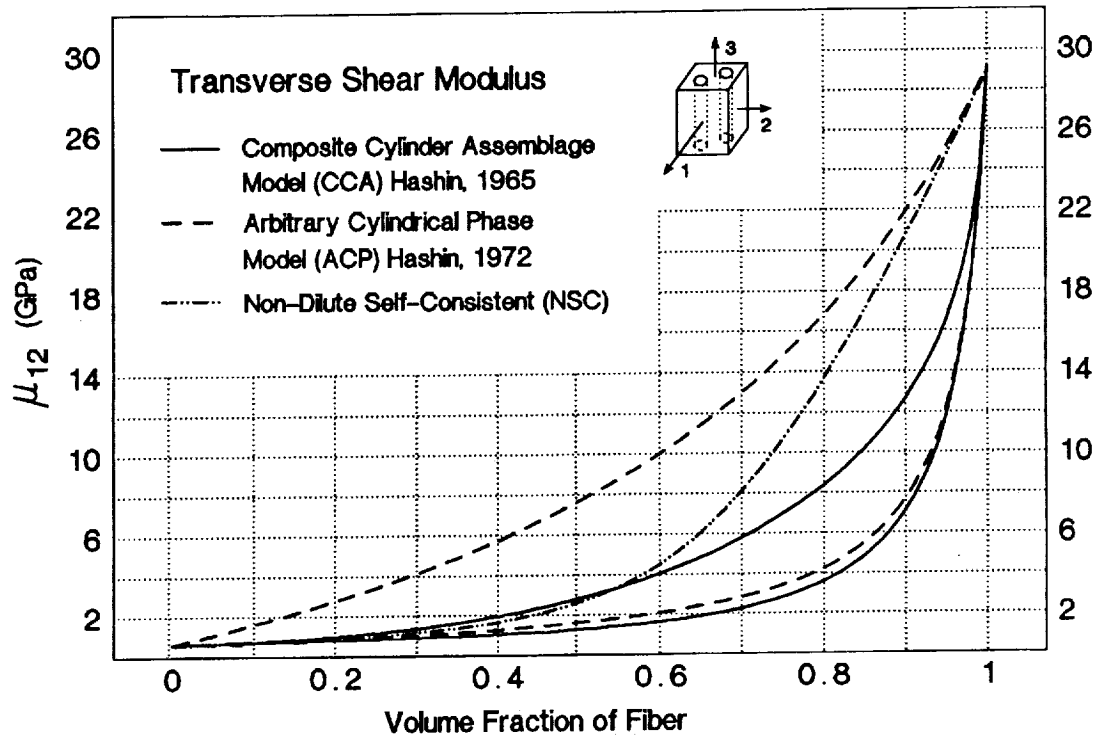


Fig. 6. Comparison of both Sets of Hashin Bounds for a Glass/Polyester Fiber Composite with the non-dilute self-consistent method.



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16. Abstract The Walker et al. (1989) self-consistent method to predict both the elastic and inelastic effective material properties of composites is examined and compared with the results of other self-consistent and elasticity based solutions. The elastic part of their method is shown to be identical to other self-consistent methods for non-dilute reinforced composite materials; they are the Hill (1965), Budiansky (1965) and Nemat-Nasser et al. (1982) derivations. A simplified form of the non-dilute self-consistent method is also derived. The predicted, effective, elastic, material properties for a fiber reinforced material using the Walker method was found to deviate from the elasticity solution for the ν_{31} , K_{12} , and μ_{31} material properties (fiber is in the 3 direction) especially at the larger volume fractions. Also, the prediction for the transverse shear modulus, μ_{12} , exceeds one of the accepted Hashin bounds. Only the longitudinal elastic modulus E_{33} agrees with the elasticity solution. The differences between the Walker and elasticity solutions are primarily due to the assumption used in the derivation of the self-consistent method, i.e. the strain fields in the inclusions and the matrix are assumed to remain constant, which is not a correct assumption for a high concentration of inclusions.			
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